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Confining solutions of $(n+1)$-dimensional Yang-Mills equations for flat and curved space-time with $n \leq 3$

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Abstract

We obtain exact static solutions of the $(n+1)$-dimensional $SU(3)$ Yang-Mills equations for both flat and curved space-time cases with $n \leq 3$. We find that the solutions obtained are confining functions for $n = 1, 2, 3$. We apply the $(3+1)$ curved space-time solution to the anti-de Sitter and Schwarzschild metrics.

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1 Introduction

The quark confinement problem [1] can not be directly solved by the Quantum Chromodynamics (QCD) due the perturbation theory fails in the infrared regime [2]. As an alternative the color charge of the $SU(3)$ gauge fields has been related to quark confinement [3]. The semiclassical approach permits to obtain non-perturbative information about QCD starting from the solutions of the classical partial differential equation of the $SU(3)$ Yang-Mills theory [4]. To understand the role of non-abelian gauge fields in the quark confinement problem has motivated the search for solutions of the classical Yang-Mills field equations in presence of static external sources. One of the first works about this subject showed that if the external source is distributed over a thin spherical shell the Coulomb solution is unstable in a specific regime [3]. A wide range of solutions of $(3+1)$-dimensional Yang-Mills fields equations in presence of localized and extended external sources were obtained later [4]-[8]. As well, some specific solutions of the $(2+1)$-dimensional $SU(2)$ Yang-Mills equations were also obtained [9]. On the other hand the discovery of globally regular solutions of the Einstein-Yang-Mills equations with $SU(2)$ gauge group [10] originated a great interest about spherical symmetric solutions [11]. Additionally, the study of the quark confinement problem in curved space-time has been a subject of interest [12].

A semiclassical approach motivated in the black holes physics technics was recently proposed to describe the energy spectra of quarkonia by solving the Dirac equation in presence of $SU(3)$ Yang-Mills fields representing gluonic fields [13]. In the context of this approach, explicit calculations have shown how gluon concentration is huge at scales of the order of 1 fm [14]. The obtained solutions can model the quark confinement in a satisfactory way suggesting that the mechanism of quark confinement should occur within the framework of QCD [15]. This fact implies that the gluon fields form a boson condensate and, therefore, gluons can be considered as classical fields [15]. By this reason the dynamics of the strong interaction at large distances would be described by the equations of motion of the $SU(3)$ Yang-Mills theory [15].

Following this semiclassical approach, presented in detail in [15], we obtain exact static solutions of the $(n+1)$-dimensional $SU(3)$ Yang-Mills equations for both flat and curved space-time cases with $n \leq 3$. We find, in both cases, confining solutions for $n = 1, 2, 3$. As an application of the $(3+1)$ curved space-time solution we consider the anti-de Sitter and Schwarzschild metrics.

2 Preliminaries

We work in the Minkowski space M in which the line element is given by

$$ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu, \tag{2. 1}$$

where the components $g_{\mu\nu}$ take different values depending on the choice of coordinates and dimensions. The *Hodge start operator* $*$ is defined as: $\Lambda^p(M) \to \Lambda^{n-p}(M)$, where $\Lambda^p(M)$ is the *p-form* over a differentiable variety M of dimension n. If $\{dx^1, ..., dx^n\}$ is the base for $\Lambda^p(M)$ then

$$*(dx^{i_1} \wedge ... \wedge dx^{i_p}) = \frac{g^{(1/2)}}{(n-p)!} g^{i_1 l_1} ... g^{i_p l_p} \varepsilon_{l_1 ... l_p l_{p+1} ... l_n} dx^{l_{p+1}} \wedge ... \wedge dx^{l_n}. \tag{2. 2}$$

The *exterior differential* d is defined as: $\Lambda^p(M) \to \Lambda^{p+1}(M)$. This means that

$$d = \partial_\mu dx^\mu. \tag{2. 3}$$

Let $A = A_\mu dx^\mu = A^a_\mu \lambda_a dx^\mu$ be a connection in $SU(N)$, where λ_a are generators of the $SU(N)$ gauge group, with $a = 1, 2, ..., N^2 - 1$, and A^a_μ the non-abelian fields. The Yang-Mills equations for $SU(N)$ can be written using the Hodge star operator as

$$d * F = g(*F \wedge A - A \wedge *F) + gJ, \tag{2. 4}$$

where $F = dA + A \wedge A = F^a_{\mu\nu} \lambda_a dx^\mu \wedge dx^\nu$ is the curvature and g is the coupling constant. The non-abelian $SU(N)$ current density J is given by

$$J = j^a_\mu \lambda_a * (dx^\mu) = *j = *(j^a_\mu \lambda_a dx^\mu). \tag{2. 5}$$

As an example, this current density for the case of a point particle at rest is $J = j^a_\mu \lambda_a dx^\mu = \delta(\vec{r}) q^a \lambda_a dt$, where q^a are constants and then $q^a \lambda_a = \Upsilon$ is a constant.

In a similar way as in the functional quantization of Yang-Mills theories, we fix the gauge through the condition $div(A) = 0$, or similarly

$$div(A) = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} g^{\mu\nu} A_\nu) = 0. \tag{2. 6}$$

3 Solutions in a flat $(n+1)$-dimensional space-time

The $SU(3)$ Yang-Mills equations are a non-linear system of coupled partial differential equations. In this section we present some exact static solutions of the $SU(3)$ Yang-Mills equations in a flat space-time of $(n+1)$ dimensions with $n \leq 3$, following the same techniques used in [15]. We consider the cases $n = 1, 2, 3$ and we find that in the three cases the solutions are confining functions.

3.1 Case $n = 1$

We first consider the Yang-Mills equations in (1+1) dimension using the Minkowskian metric given by

$$ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu = dt^2 - dx^2. \tag{3. 1}$$

We assume that A has the form $A = A_\mu(x)dx^\mu = A_t(x)dt + A_x(x)dx$, where $A_t(x)$ is a function on x (that we call $f(x)$) that is written as a linear combination of the gauge group generators. For this case, the gauge condition (2. 6) leads to

$$\partial_x(A_x(x)) = 0, \tag{3. 2}$$

being $A_x(x)$ a constant. In this coordinates, the exterior differential is written as

$$d = \partial_t dt + \partial_x dx. \tag{3. 3}$$

For this case the Yang-Mills equations (2. 4) lead to

$$\partial_x^2(A_t(x)) = \delta(x) q^a \lambda_a = \delta(x)\Upsilon, \tag{3. 4}$$

where we have taked $A_x = 0$ and we have used the fact that $*(dx) = dx$. The solution of the last equation is given by $f(x) = l + k|x|$, where l and k are constants. This solution is invariant under parity transformation and is a linear confining solution.

3.2 Case $n = 2$

Now we consider the Yang-Mills equations in (2+1) dimension. We use the Minkowskian metric in polar coordinates

$$ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu = dt^2 - dr^2 - r^2 d\theta^2. \tag{3. 5}$$

If we suppose the solutions of the form $A_\theta(r, \theta) = g(r)\theta + h(r)$, it is possible to see that A_θ does not depend on θ. In this case the gauge condition (2. 6) leads to the following equation

$$\partial_r(rA_r(r)) = 0, \tag{3. 6}$$

being its solution given by $A_r = C/r$. If we assume that $C = 0$, we can write A in terms of $A_t = f(r)\Gamma$ and $A_r = g(r)\Delta$, being Γ and Δ linear combinations of the group generators. As the exterior differential, in polar coordinates, has the form

$$d = \partial_t dt + \partial_r dr + \partial_\theta d\theta, \tag{3. 7}$$

and the following relations are satisfied

$$
\begin{aligned}
*(dt \wedge dr) &= -r d\theta, \\
(dt \wedge d\theta) &= \frac{1}{r} dr, \\
(dr \wedge d\theta) &= \frac{1}{r} dt,
\end{aligned}
$$

then the curvature is given by

$$F = dA + gA\wedge A = -\partial_r f(r)\Gamma dt \wedge dr + \partial_r g(r)\Delta dr \wedge d\varphi + g f(r) g(r)[\Gamma, \Delta] dt \wedge d\theta, \tag{3. 8}$$

and the Hodge star operator applied over the curvature is

$$*F = r\partial_r f(r)\Gamma d\theta + \frac{\partial_r g(r)}{r}\Delta dt + g\frac{f(r)g(r)}{r}[\Gamma, \Delta] dr. \tag{3. 9}$$

If we demand that $[\Gamma, \Delta] = 0$ and we consider that $*dt = r dr \wedge d\theta$, we obtain the following differential equation system

$$\partial_r(\frac{1}{r}\partial_r g(r))\Delta = 0, \tag{3. 10}$$

$$\partial_r(r\partial_r f(r))\Gamma = r\delta(r)\Upsilon. \tag{3. 11}$$

The solutions of this equation system is given by $f(r) = d + k\log r$ and $g(r) = d + kr^2$. The function $f(r)$ is the known confining solution in the two dimension problem [16]. We observe that $g(r)$ is also a confining function. It is clear that in this case there exists confinement and this fact is associated separately with the temporal and spatial parts of the gluon fields. The abelian condition $[\Gamma, \Delta] = 0$ is satisfied in a not trivial way if and only if one of the following conditions holds:

1) If in the combinations of Γ and Δ, in terms of the group generators, only appears the matrix which constitute the Cartan subalgebra of the $SU(N)$-Lie algebra.

2) If $\Gamma = k\Delta$, being k a constant.

3.3 Case $n = 3$

The solutions in spherical coordinates of the $SU(3)$ Yang-Mills equations were obtained in detail in [13, 14, 15]. We only present here the solution of A given by

$$A = A_t dt + A_\varphi d\varphi = (B + \frac{b}{r})\Gamma dt + (C + cr)\Delta d\varphi, \tag{3. 12}$$

which is obtained using the abelian condition $[\Gamma, \Delta] = 0$. We note that this solution corresponds to the Cornell potential [17] which has been used to describe phenomenological features of QCD, as this is a confining function.

4 Solutions in a curved $(n+1)$-dimensional space-time

In this section we present some static and exact solutions of the $SU(3)$ Yang-Mills equations in a curved space-time of $(n+1)$ dimensions with $n \leq 3$. We consider independently the $n = 1, 2, 3$ cases. As in the past section, the solutions for the three cases are confining functions.

4.1 Case $n = 1$

We consider a curved space-time in $(1+1)$ dimensions which metric is given by

$$ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu = \alpha^2(x) dt^2 - \beta^2(x) dx^2. \tag{4. 1}$$

We assume, as in the flat $(1+1)$ space-time case, that A has the following functional dependence

$$A = A_\mu(x) dx^\mu = A_t(x) dt + A_x(x) dx = \lambda_a f^a(x) dt + A_x(x) dx, \tag{4. 2}$$

where λ_a are the group generators and the number of functions $f^a(x)$ is the same as the number of generators. The gauge condition (2. 6) implies that

$$\partial_x \left(\frac{\alpha(x) A_x(x)}{\beta(x)} \right) = 0, \tag{4. 3}$$

and then $A_x = C\beta(x)/\alpha(x)$. The exterior differential is given by (3. 3). Putting $C = 0$ and using that $*(dt \wedge dx) = \frac{1}{\alpha(x)\beta(x)}$, we obtain

$$*F = -\frac{\lambda_a \partial_x f^a(x)}{\alpha(x)\beta(x)}. \tag{4. 4}$$

For this case we have that $*(dt) = \frac{\beta(x)}{\alpha(x)} dx$ and the expression (2. 4) leads to

$$\partial_x \left(\frac{\lambda_a \partial_x f^a(x)}{\alpha(x)\beta(x)} \right) dx = g \left(\frac{\partial_x f^a(x)}{\alpha(x)\beta(x)} f^b(x) [\lambda_a, \lambda_b] \right) dt + \delta(x) \frac{\beta(x)}{\alpha(x)} q^a \lambda_a dx. \tag{4. 5}$$

To have a non-trivial solution of this equation is necessary that $[\lambda_a, \lambda_b] = 0$ or alternatively $\lambda_a \partial_x f^a(x) = f'(x)\Gamma$, where Γ is a linear combination of the group generators. Then we obtain the following equation

$$\partial_x \left(\frac{\lambda_a \partial_x f^a(x)}{\alpha(x)\beta(x)} \right) = \delta(x) \frac{\beta(0)}{\alpha(0)} q^a \lambda_a, \tag{4. 6}$$

whose solution is given by

$$f^a(x) = k^a \int \alpha(x)\beta(x)dx + d^a, \tag{4. 7}$$

where k^a and d^a are constants. For the case in which $\alpha = \beta = 1$ we obtain the solution of the flat space-time case, i. e. $f(x) = d + k|x|$.

4.2 Case $n = 2$

We consider now a curved space-time in $(2+1)$ dimensions given by

$$ds^2 = g_{\mu\nu}dx^\mu \otimes dx^\nu = \alpha^2(r)dt^2 - \beta^2(r)dr^2 - r^2 d\theta^2. \tag{4. 8}$$

We assume that A has the following functional dependence:

$$A = A_\mu(r)dx = A_t(r)dt + A_r(r)dr + A_\theta(r,\theta)d\theta. \tag{4. 9}$$

The gauge condition (2. 6), for this case, leads to

$$\partial_r\left(\frac{r\alpha(r)A_r(r)}{\beta(r)}\right) + \partial_\theta\left(\frac{\alpha(r)\beta(r)A_\theta(r,\theta)}{r^2}\right) = 0. \tag{4. 10}$$

If we discard the solutions of the form $A_\theta(r,\theta) = g(r)\theta + h(r)$, we can see that A_θ does not depend on θ and then the gauge condition (2. 6) leads to the equation

$$\partial_r\left(\frac{r\alpha(r)A_r(r)}{\beta(r)}\right) = 0, \tag{4. 11}$$

whose solution has the form $A_r = C\frac{\beta(r)}{r\alpha(r)}$. The exterior differential is given by (3. 7). We put $C = 0$ and assume that A can be written as $A_t = f(r)\Gamma$ and $A_r = g(r)\Delta$, being Γ and Δ linear combinations of the group generators. If we also consider the following conditions

$$\begin{aligned}
*(dt \wedge dr) &= -\frac{r}{\alpha(r)\beta(r)}d\theta, \\
(dt \wedge d\theta) &= \frac{\alpha(r)\beta(r)}{r}dr, \\
(dr \wedge d\theta) &= \frac{\alpha(r)}{r\beta(r)}dt,
\end{aligned}$$

we obtain that the curvature has the same form as the one of the flat space-time case, which is given given by (3. 8), i. e.

$$*F = \frac{r\partial_r f(r)\Gamma}{\alpha(r)\beta(r)}d\theta + \frac{\alpha(r)}{r\beta(r)}\frac{\partial_r g(r)}{r}\Delta dt + g\frac{\alpha(r)\beta(r)f(r)g(r)}{r}[\Gamma,\Delta]dr. \tag{4. 12}$$

Using the expression (2. 4), we have that

$$\partial_r\left(r\frac{\partial_r f(r)}{\alpha(r)\beta(r)}\Gamma\right)dr\wedge d\theta-\partial_r\left(\frac{\alpha(r)\partial_r g(r)}{r\beta(r)}\Delta\right)dt\wedge dr \;=\; \delta(\vec{r})\frac{r\beta(r)}{\alpha(r)}\Upsilon dr\wedge d\theta$$
$$+\;\; g^2\left(\frac{f(r)g(r)^2\alpha(r)\beta(r)}{r}[[\Gamma,\Delta],\Delta]dr\wedge d\theta-\frac{f(r)^2g(r)\alpha(r)\beta(r)}{r}[[\Gamma,\Delta],\Gamma]dt\wedge dr\right). \tag{4. 13}$$

If we demand that $[\Gamma,\Delta]=0$, equation (4. 13) leads to the following equation system

$$\partial_r\left(\frac{\alpha(r)}{r\beta(r)}\partial_r g(r)\right) \;=\; 0, \tag{4. 14}$$

$$\partial_r\left(\frac{r}{\alpha(r)\beta(r)}\partial_r f(r)\right) \;=\; \delta(\vec{r})\frac{r\beta(0)}{\alpha(0)}\Upsilon. \tag{4. 15}$$

The solution of this equation system is given by

$$g(r) \;=\; k_2\int\frac{r\beta(r)}{\alpha(r)}dr+d_2, \tag{4. 16}$$

$$f(r) \;=\; k_1\int\frac{\alpha(r)\beta(r)}{r}dr+d_1. \tag{4. 17}$$

For the case $\alpha=\beta=1$ we obtain the solution of the $(2+1)$ flat space-time case, i. e. $g(r)=d_2+k_2r^2$ and $f(r)=d_1+k_1\log r$.

4.3 Case $n=3$

The metric for a curved static space-time and spherically symmetric can be specified by

$$ds^2=g_{\mu\nu}dx^\mu\otimes dx^\nu=\alpha^2(r)dt^2-\beta^2(r)dr^2-r^2(d\theta^2+sin^2\theta d\varphi^2). \tag{4. 18}$$

We assume that A has the following functional dependence

$$A=A_\mu(r)dx^\mu=A_t(r)dt+A_r(r)dr+A_\theta(r)d\theta+A_\varphi(r)d\varphi. \tag{4. 19}$$

For this case, the gauge condition (2. 6) leads to

$$\partial_r\left(\frac{r^2\alpha(r)A_r(r)}{\beta(r)}\right)+\alpha(r)\beta(r)A_\theta(r)cot\theta=0, \tag{4. 20}$$

and then, $A_\theta=0$ and $A_r=C\frac{\beta(r)}{r^2\alpha(r)}$. We can put $C=0$ in the A_r solution because this does not affect the form of A_t and A_φ solutions. Now we take $A_t=f(r)\Gamma$ and

$A_\varphi = g(r)\Delta$, being Δ and Γ linear combination of the group generators. As the exterior differential in spherical coordinates is given by

$$d = \partial_t dt + \partial_r dr + \partial_\theta d\theta + \partial_\varphi d\varphi, \tag{4. 21}$$

then the curvature is given by

$$F = dA + gA\wedge A = -\partial_r f(r)\Gamma dt\wedge dr + \partial_r g(r)\Delta dr\wedge d\varphi + gf(r)g(r)[\Gamma,\Delta]dt\wedge d\varphi. \tag{4. 22}$$

Applying the Hodge start operator (2. 4) over F and using the following relations

$$\begin{aligned}
*(dt\wedge dr) &= -\frac{r^2 sin\theta}{\alpha(r)^2\beta(r)^2}d\theta\wedge d\varphi,\\
(dt\wedge d\theta) &= \frac{sin\theta}{\alpha(r)^2}dr\wedge d\varphi,\\
(dt\wedge d\varphi) &= \frac{-1}{\alpha(r)^2 sin\theta}dr\wedge d\theta,\\
(dr\wedge d\varphi) &= \frac{-1}{\beta(r)^2 sin\theta}dt\wedge d\theta,\\
(dr\wedge d\theta) &= \frac{sin\theta}{\beta(r)^2}dt\wedge d\varphi,\\
(d\theta\wedge d\varphi) &= \frac{1}{r^2 sin\theta}dt\wedge dr,
\end{aligned}$$

we obtain, for $r\neq 0$, the following equation system

$$\partial_r\left(\frac{\partial_r g(r)}{\beta(r)^2}\right)\Delta = \frac{g^2}{\alpha(r)^2}f(r)^2 g(r)[\Gamma,[\Gamma,\Delta]], \tag{4. 23}$$

$$\partial_r\left(\frac{r^2\partial_r f(r)}{\alpha(r)^2\beta(r)^2}\sin^2\theta\right)\Gamma = \frac{g^2 f(r)g(r)^2}{\alpha(r)^2}[\Delta,[\Gamma,\Delta]]. \tag{4. 24}$$

As for the flat space-time case, we demand this equation system to satisfy the condition $[\Delta,[\Gamma,\Delta]] = 0$ and then we obtain

$$\partial_r\left(\frac{\partial_r g(r)}{\beta(r)^2}\right) = 0, \tag{4. 25}$$

$$\partial_r\left(\frac{r^2\partial_r f(r)}{\alpha(r)^2\beta(r)^2}\right) = 0. \tag{4. 26}$$

The solutions of these two equation are

$$g(r) = b_1 \int \beta(r)^2 dr + B_1, \qquad (4.\ 27)$$

$$f(r) = a_1 \int \frac{\alpha(r)^2 \beta(r)^2}{r^2} dr + A_1. \qquad (4.\ 28)$$

We observe that for $\alpha = \beta = 1$, the solutions (4. 27) and (4. 28) allow to the confining solutions of the flat space-time case given by $+g(r) = b_1 r + B_1$ and $f(r) = -a_1/r + A_1$.

As a particular application of the $(3+1)$ case solution we consider now the anti-de Sitter metric given by

$$ds^2 = (1 - \Lambda r^2/3)dt^2 - (1 - \Lambda r^2/3)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2). \qquad (4.\ 29)$$

Because $\alpha(r)$ is the inverse of $\beta(r)$, then the Coulomb solution has not deformations respect to the flat space-time case and it is given by $f(r) = a_1/r + A_1$. The linear solution $g(r)$ changes respect to the flat case. We obtain the solution explicitly in the the two following situations $\Lambda > 0$ and $\Lambda < 0$, so

$$g(x) = \begin{cases} b_1 \tanh^{-1}\left(\frac{r\Lambda^{1/2}}{3^{1/2}}\right) + B_1, & \text{if } \Lambda > 0, \\ b_1 \tan^{-1}\left(\frac{r(-\Lambda)^{1/2}}{3^{1/2}}\right) + B_1, & \text{if } \Lambda < 0, \end{cases} \qquad (4.\ 30)$$

The function $g(r)$, for the limit cases $|\Lambda| << 1$ and $r << 1$, has the form $g(r) \simeq b_1 r + B_1$, recovering the the flat space-time case behavior.

Another application for the $(3+1)$ curved space-time solution is the Schwarzschild metric given by

$$ds^2 = (1 - 2M/r)dt^2 - (1 - 2M/r)^{-1} dr^2 - r^2(d\theta^2 + sin^2\theta d\varphi^2). \qquad (4.\ 31)$$

As happened in the first application, the Coulomb solution has not deformations respect to the flat space-time case, but the linear solution has. The function $g(r)$ for this case is

$$g(r) = b_1(r + 2M \ln|r - 2M|) + B_1 \qquad (4.\ 32)$$

This solution, for the limit $r >> M$, has the form $g(r) \simeq b_1 r + B_1$, recovering the flat space-time case behavior.

5 Summary

We have presented some exact static solutions for the $SU(3)$ Yang-Mills equations in a flat and a curved space-time of $(n+1)$ dimensions with $n \leq 3$. We have found in

both cases there are confining solutions for $n = 1, 2, 3$. For the $(1+1)$ case, both in the flat and curved space-time, we found that the solution for the temporary part can be written as $A_t = f(r)\Gamma$. To find analytic solutions in the $(2+1)$ case is necessary to demand the abelian condition given by $[\Delta, \Gamma] = 0$. For the cases (1+1) and (3+1) this condition is satisfied naturally. We presented in detail the solution for $(3+1)$ curved space-time case and we applied this solution to the anti-de Sitter and Schwarzschild cases. In both cases the Coulomb solution does not have deformations respect to the flat space-time case, while in the linear solution there exists deformation. As a perspective it would be interesting to understand the role of the confining solutions in a model of relativistic quark confinement in low dimensionality.

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References

[1] A. Chodos, R. L. Jaffe, K. Johnson, C. B. Thorn and V. F. Weisskopf, Phys. Rev. D9 (1974) 3471; Phys. Rev. D10 (1974) 2599; K. G. Wilson, Phys. Rev. D10 (1974) 2445; Y. Nambu, Phys. Rev. D10 (1974) 4262; C. G. Callan, R. Dashen and D. J. Gross, Phys. Rev. D17 (1978) 2717.

[2] W. J. Marciano and H. Pagels, Phys. Rept. 36 (1978) 137.

[3] J. E. Mandula, Phys. Rev. D14 (1976) 3497.

[4] R. Jackiw, L. Jacobs and C. Rebbi, Phys. Rev. D20 (1979) 474; R. Jackiw and P. Rossi, Phys. Rev. D21 (1980) 426.

[5] P. Sikivie and N. Weiss, Phys. Rev. Lett. 40 (1978) 1411; Phys. Rev. D18 (1978) 3497; Phys. Rev. D20 (1979) 487.

[6] D. Horvat and K. S. Viswanathan, Phys. Rev. D23 (1981) 937; D. Horvat, Phys. Rev. D34 (1986) 1197.

[7] R. Teh, W. K. Koo and C. H. Oh, Phys. Rev. D23 (1981) 3046; C. H. Oh, R. Teh and W. K. Koo, Phys. Rev. D24 (1981) 2305; Phys. Rev. D25 (1982) 3263;

C. H. Oh, Phys. Rev. D25 (1982) 2194; J. Math. Phys. 25 (1984) 660; Phys. Rev. D47 (1993) 1652; C. H. Oh, S. N. Show and C. H. Lai, Phys. Rev. D30 (1984) 1334; C. H. Oh, C. H. Lai and C. P. Soo, Phys. Rev. D32 (1985) 2843.

[8] K. Cahill, Phys. Rev. Lett. 41 (1978) 599; U. Sarkar and Raychaudhuri, Phys. Rev. D26 (1982) 2804; H. Arodz, Nucl. Phys. B207 (1982) 288; Acta Phys. Polon. B14 (1983) 825; Phys. Rev. D35 (1987) 4024; F. Nill, Ann. Phys. (N.Y.) 149 (1983) 303; G. K. Savvidy, Phys. Lett. B130 (1983) 303; S. J. Chang, Phys. Rev. D29 (1984) 259; S. K. Paul and A. Khare, Phys. Lett. B138 (1984) 402; D. Sivers, Phys. Rev. D34 (1986) 1141; E. Malec, Acta Phys. Pol. B18 (1987) 1017.

[9] E. D'Hoker and L. Vinet, Ann. Phys. (N.Y.) 162 (1985) 413; C. H. Oh, L. H. Sia and R. Teh, Phys. Rev. D40 (1989) 601.

[10] R. Bartnik and J. McKinnon, Phys. Rev. Lett. 61 (1988) 141.

[11] P. Bizon, Phys. Rev. Lett. 64 (1990) 2844; M. S. Volkov and D. V. Gatsov, Phys. Rept. 319 (1999)1; E. Winstanley, Class. Quant. Grav. 16 (1999) 1963; Z. Zou, J. Math. Phys. 42 (2001) 1085; A. G. Wasserman, J. Math. Phys. 41 (2000) 6930; Y. Brihaye, A. Chakrabarti and D. H. Tchrakian, Class. Quant. Grav. 20 (2003) 2765; Y. Brihave and B. Hartmann, Class. Quant. Grav. 22 (2005) 183; Y. Brihaye, E. Redu and D. H. Tchrahian, Phys. Rev. D75 (2007) 024022.

[12] C. C. Barros Jr., Eur. Phys. J. C45 (2006) 421.

[13] Y. P. Goncharov, Mod. Phys. Lett. A16 (2001) 557; Europhys. Lett. 62 (2003) 684; Y. P. Goncharov and E. A. Choban, Mod. Phys. Lett. A18 (2003) 1661; arXiv:hep-th/0512099 v1 8 Dec 2005; Phys. Lett. B641 (2006) 237.

[14] Y. P. Goncharov and A. A. Bytsenko, Phys. Lett. B602 (2004) 86.

[15] Y. P. Goncharov, Phys. Lett. B 617 (2005) 67.

[16] G. B. Arfken and H. J. Weber, *Mathematical methods for physicists*, fifth edition, Academic Press (2001).

[17] E. Eichten, K. Gottfried, T. Kinoshita, K. D. Lane, and T. M. Yan, Phys. Rev. D17 (1978) 3090.